Question One **Marks**

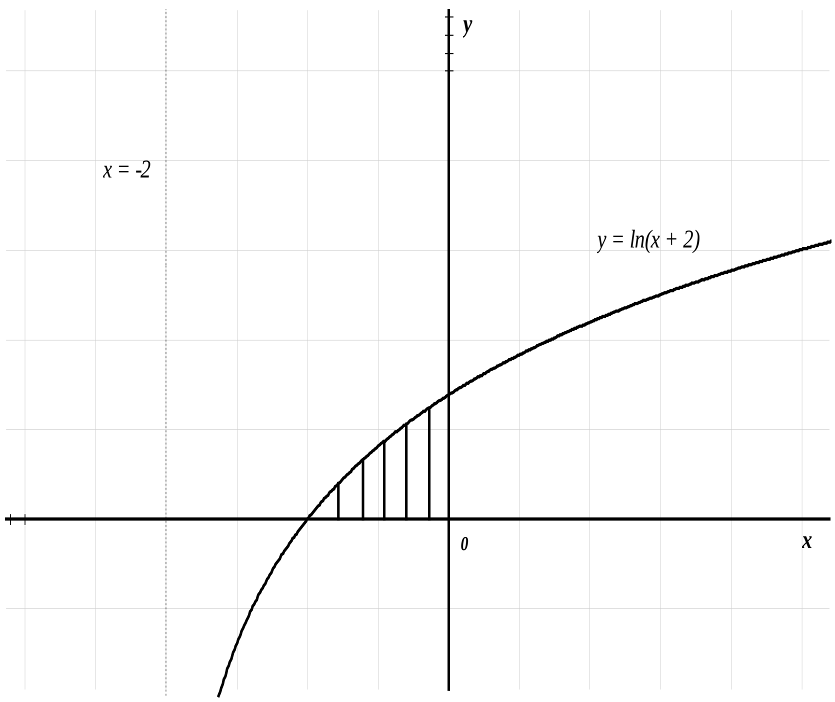
a) Find $\int sin(4x+6)dx$. **1**

b) Differentiate 5^x. **1**

c) For what values of x will $1-tan^2 x+tan^4 x-tan^6 x+tan^8 x-...$ **4**

have a limiting sum for $0 \le x \le 2\pi$.

d) The sketch of the curve of $y = ln(x+2)$ is shown below. **4**
If the shaded area is rotated about the y-axis, find the volume of revolution of the solid generated.



Question Two (Start a New Page)

a) i) Express $y = \sqrt{3}\cos x - \sin x$ in the form of $R\cos(x-\alpha)$, **3**
where $R \rangle 0$ and $0 \le \alpha \le 2\pi$.

ii) Sketch the graph of $y = \sqrt{3}\cos x - \sin x$, for $0 \le x \le 2\pi$. **3**

b) i) Differentiate $ln(\sin x) - x\cot x$. **2**

ii) Hence find $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x(cosec^2 x)dx$. **2**

Question Three (Start a New Page) **Marks**

a) Prove that $2^{10n+3} + 3$ is divisible by 11 for all non-negative integers n by Mathematical Induction. **5**

b) A spherical map of the earth is being inflated at a constant rate of $25\text{cm}^3\text{s}^{-1}$. Find the rate at which the length of the equator is changing when the radius is 10cm. **5**

Question Four (Start a New Page)

a) Differentiate $ln\left(\dfrac{\sqrt{x}}{x+4}\right)$. **2**

b) Consider the function $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$,

i) Prove that y has no stationary points. **2**

ii) Prove that the lines $y = \pm 1$ are asymptotes. **2**

iii) If k is a positive constant, find the area in the first quadrant enclosed by the above curve and the three lines $y=1$, $x=0$ and $x=k$. **3**

iv) Prove that for all values of k, the area is always less than $ln\ 2$. **1**

Question Five (Start a New Page)

a) Find $\int \dfrac{x-1}{x+5}\,dx$. **2**

b) Evaluate $\displaystyle\int_0^{ln\sqrt{3}} \frac{e^x}{1+e^{2x}}\,dx$. **3**

c) Amy borrows \$130 000 to start a sign writing business. Interest is charged on the balance owing at the rate of 9% per annum, compounded monthly. Amy agrees to repay the loan, including interest, by making equal monthly instalment of \$$P$.

i) How much does Amy owe at the end of the first month just before she makes an instalment payment? **1**

Question Five cont'd **Marks**

c) ii) Show that if the loan is repaid after *n* months, then **1**

$$P = \frac{130000(1.0075)^n}{1+1.0075+1.0075^2+...+1.0075^{n-1}}.$$

iii) Calculate how many months, to the nearest month, it will take for the loan to be repaid if Amy makes instalments of \$1800 per month. **3**

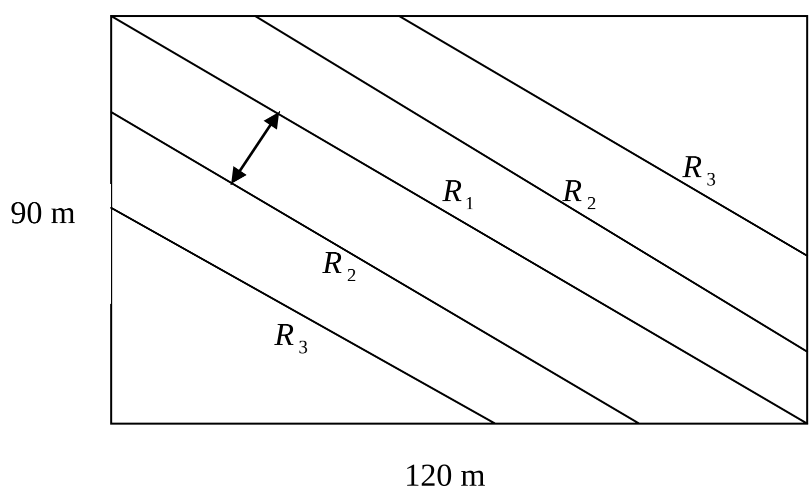
Question Six (Start a New Page)

a) Using the substitution $x = 2\sin\theta$, show by integration that **4**

$\int\sqrt{4-x^2}\ dx = \frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2} + C$, where C is a constant.

b) A rectangular paddock in a vineyard measures 90m by 120m. In order to make best use of the sun, the grape vines are planted in diagonal rows as shown, with a 3 metre gap between adjacent rows.

Diagram not to scale



i) Find the length of R_1, the diagonal of the field. **1**

ii) Show that length of the R_2 is 143.75 m. **2**

iii) Given that the rows $R_1 + R_2 + R_3 + R_4 + \ldots$ form an arithmetic series, find the total number of rows of vines in the paddock. **3**

END OF PAPER

Yr 12 (Ass 2) Term 1 2007 — Ext I

Question One

a) $\frac{-\cos(4x+b)}{4} + c$

b) $5^x(\ln 5)$ 1

c) $r = -\tan^2 x$ 1

$|r|<1$ when $\tan^2 x < 1$

$-1 < \tan x < 1$ 1

$\therefore 0 < x < \frac{\pi}{4}$, $\frac{3\pi}{4} < x < \frac{5\pi}{4}$, $\frac{7\pi}{4} < x < 2\pi$, $x \neq \pi$ 2

d) vol of revolution (V)

when $x=0$, $y = \ln 2$

$y = \ln(x+2)$

$x = e^y - 2$

$= \int_0^{\ln 2} \pi x^2 \, dy$ 1 mark for correct limit

$= \pi \int_0^{\ln 2} (e^y - 2)^2 dy$ 1

$= \pi \int_0^{\ln 2} \left[e^{2y} - 4e^y + 4\right] dy$

$= \pi \left[\frac{e^{2y}}{2} - 4e^y + 4y\right]_0^{\ln 2}$ 1

$= \pi\left[\frac{4}{2} - 4.2 + 4\ln 2\right] - \pi\left[\frac{1}{2} - 4\right]$

$= \pi\left[4\ln 2 - 2\frac{1}{2}\right]$ unit^3 # 1

Question Two

a i) $R\cos(x-\alpha) = R[\cos x \cos\alpha + R\sin x . \sin\alpha]$

$= \sqrt{3}\cos x - \sin x$

$R>0$, $\left.\begin{array}{l}\sqrt{3} = R\cos\alpha \\ -1 = R\sin\alpha\end{array}\right\} \therefore \alpha$ in 4^{th} Quad

$\tan\alpha = -\frac{1}{\sqrt{3}}$ 1 $\alpha = \frac{11\pi}{6}$ 1

$(R\cos\alpha)^2 + (R\sin\alpha)^2 = (\sqrt{3})^2 + (-1)^2$

$R^2 = 4 \quad \therefore R = 2 \quad (R>0)$

$\therefore \sqrt{3}\cos x - \sin x = 2\cos\left(x - \frac{11\pi}{6}\right)$ 1

$\therefore y = 2\cos\left(x - \frac{11\pi}{6}\right)$ Period : 2π

At $x=0$ $y = 2\cos\left(-\frac{11\pi}{6}\right) = \sqrt{3}$

$y=0$ when $\cos\left(x - \frac{11\pi}{6}\right) = 0$

$x - \frac{11\pi}{6} = -\frac{\pi}{2}$ $\therefore x = \frac{4\pi}{3}$

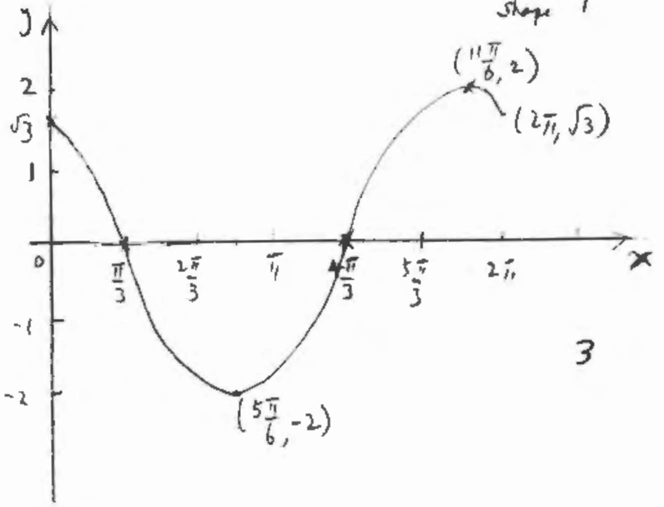
or $x - \frac{11\pi}{6} = -\frac{3\pi}{2}$ $x = \frac{\pi}{3}$

MAX at $y=2$ $2 = 2\cos\left(x - \frac{11\pi}{6}\right)$

$0 = x - \frac{11\pi}{6}$

$x = \frac{11\pi}{6}$

MAX/MIN 1
x, y intercepts 1
Shape 1



3

Let $y = \ln(\sin x) - x\cot x$

b i) $y' = \frac{\cos x}{\sin x} + x\,\text{cosec}^2 x - \cot x$ 1

$= \cancel{\cot x} + x\,\text{cosec}^2 x - \cancel{\cot x}$

$y' = x\,\text{cosec}^2 x$ # 1

ii) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x\,\text{cosec}^2 x \, dx = \left[\ln|\sin x| - x\cot x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$= \left(\ln\left|\sin\frac{\pi}{2}\right| - \frac{\pi}{2}\cot\frac{\pi}{2}\right) - \left(\ln\left|\sin\frac{\pi}{6}\right| - \frac{\pi}{6}\cot\frac{\pi}{6}\right)$

$= (\cancel{\ln 1} - \cancel{\tfrac{\pi}{2} \times 0}) - \left(\ln\frac{1}{2} - \frac{\pi}{6}\sqrt{3}\right)$

$= -\ln\frac{1}{2} + \frac{\pi}{6}\sqrt{3} = \frac{\pi}{6}\sqrt{3} + \ln 2$ 1

Question 3

a) When $n=0$ $\quad 2^{10.0+3}+3 = 2^3+3 = 11$ divisible by 11 $\quad$ /

Assume $2^{10k+3}+3$ is divisible by 11 $\quad$ /

i.e. $2^{10k+3}+3 = 11Q \quad Q \in J^+$

$\therefore \quad 2^{10k+3} = 11Q - 3$

Required to prove $2^{10(k+1)+3}+3$ is divisible by 11

$2^{10(k+1)+3}+3 = \left(2^{10k+3}\right)\cdot 2^{10}+3$ $\quad$ /

$= (11Q-3)\cdot 2^{10}+3$ $\quad$ from assumption /

$= 11Q \cdot 2^{10} - 3\cdot 2^{10}+3$

$= 11Q \times 2^{10} - 3072 + 3$

$= 11Q \cdot 2^{10} - 3069$

$= 11[1024Q - 279]$ $\quad$ /

$2^{10}Q - 279$ is a positive integer

Since $Q \in J^+$

$\therefore \quad 2^{10(k+1)+3}+3$ is divisible by 11

Hence by the Principle of Mathematical Induction $2^{10+3}+3$ is divisible by all non-negative integers n.

b) $V = \frac{4}{3}\pi r^3$ (vol. of sphere)

length of equation $l = 2\pi r \quad \therefore r = \frac{l}{2\pi}$

$V = \frac{4}{3}\pi \frac{l^3}{8\pi^3} = \frac{l^3}{6\pi^2}$ $\quad$ /

$\frac{dv}{dl} = \frac{3l^2}{6\pi^2} = \frac{l^2}{2\pi^2}$ $\quad$ /

Given $\frac{dv}{dt} = 25$

$\frac{dl}{dt} = \frac{dl}{dv}\cdot\frac{dv}{dt} = \frac{1}{\frac{l^2}{2\pi^2}} \times 25 = \frac{50\pi^2}{l^2}$ $\quad$ /+1

when $r = 10 \quad l = 20\pi$

$\therefore \frac{dl}{dt} = \frac{50\pi^2}{400\pi^2} = \frac{1}{8}$ cm/s # $\quad$ /

Question 4

a) $\ln\left(\frac{\sqrt{x}}{x+4}\right) = \frac{1}{2}\ln x - \ln(x+4)$

$\frac{d}{dx}\left(\ln\frac{\sqrt{x}}{x+4}\right) = \frac{1}{2x} - \frac{1}{x+4}$ $\quad$ 2

$= \frac{x+4-2x}{2x(x+4)}$

$= \frac{4-x}{2x(x+4)}$ #

b) i) $y' = \frac{(e^x+e^{-x})(e^x+e^{-x}) - (e^x-e^{-x})(e^x-e^{-x})}{(e^x+e^{-x})^2}$ $\quad$ /

$y' = \frac{\cancel{e^{2x}}+1+1+\cancel{e^{-2x}} - (\cancel{e^{2x}}-1-1+\cancel{e^{-2x}})}{(e^x+e^{-x})^2}$

$y' = \frac{4}{(e^x+e^{-x})^2} \neq 0$ $\quad$ /

$\therefore$ No stationary point

ii) $y = \frac{(e^x - e^{-x}) \div e^x}{(e^x+e^{-x}) \div e^x}$

$y = \frac{1-e^{-2x}}{1+e^{-2x}}$

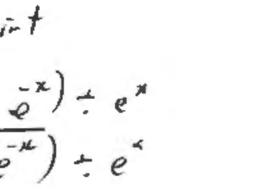
as $x \to \infty \quad e^{-2x} \to 0 \quad \therefore y \to 1$ $\quad$ /

Similarly $\quad y = \frac{(e^x - e^{-x}) \div e^{-x}}{(e^x+e^{-x}) \div e^{-x}} = \frac{e^{2x}-1}{e^{2x}+1}$

as $x \to -\infty \quad e^{2x} \to 0 \quad \therefore y \to -1$ $\quad$ /

Hence the lines $y = \pm 1$ are asymptotes

iii) Shaded area A



$= k \times 1 - \int_0^k \frac{e^x - e^{-x}}{e^x+e^{-x}}\,dx$ $\quad$ /

$= k - \ln\left[|e^x+e^{-x}|\right]_0^k$ $\quad$ /

$= k - \ln(e^k+e^{-k}) + \ln(1+1)$

$= k - \ln(e^k+e^{-k}) + \ln 2$ $\quad$ /

iv) Since $e^k > 0 \quad e^{-k} > 0$

$e^k < e^k + e^{-k}$

Taking logs on both sides $\quad k < \ln(e^k+e^{-k})$

$\therefore \quad A < \ln 2$ # $\quad$ /

Question 5

a) $\int \frac{x-1}{x+5}\,dx = \int \frac{x+5-6}{x+5}\,dx = \int 1 - \frac{6}{x+5}\,dx$

$= x - 6\ln(x+5) + c$ #

b) $\int_0^{\ln\sqrt{3}} \frac{e^x}{1+e^{2x}}\,dx = \int_1^{\sqrt{3}} \frac{1}{1+u^2}\,du$

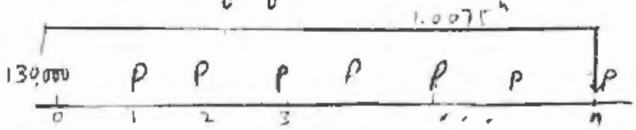
$u = e^x$
$du = e^x\,dx$
when $x = 0$
$u = 1$
when $x = \ln\sqrt{3}$
$u = \sqrt{3}$

$= \left[\tan^{-1} u\right]_1^{\sqrt{3}} = \frac{\pi}{3} - \frac{\pi}{4}$ #

c) 9% p.a = 0.75% per month

i) At the end of first month $= 130000(1.0075) - P$ #

ii)



At the end of n months the loan of \$130000 will accumulate to $130000(1.0075)^n$ and the sum of all instalments will accumulate to

$P\left[1 + 1.0075 + 1.0075^2 + \cdots + 1.0075^{n-1}\right]$

For the loan to be paid off

$130000(1.0075)^n = P\left[1 + 1.0075 + 1.0075^2 + \cdots + 1.0075^{n-1}\right]$

$\therefore P = \frac{130000\,(1.0075^n)}{1 + 1.0075 + 1.0075^2 + \cdots + 1.0075^{n-1}}$ #

iii) $1800 = \frac{130000(1.0075^n)}{1 + 1.0075 + \cdots + 1.0075^{n-1}}$ (from ii)

$1800 = \frac{130000\,(1.0075^n)}{\frac{1 - 1.0075^n}{1 - 1.0075}}$

$240000 \times (1.0075^n - 1) = 130000(1.0075^n)$

$24(1.0075^n) - 24 = 13(1.0075^n)$

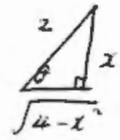
$11(1.0075^n) = 24$

$n(\log 1.0075) = \log\left(\frac{24}{11}\right)$

$n \doteqdot 104$ (nearest month)

Question 6

a) $x = 2\sin\theta$ $\quad \frac{dx}{d\theta} = 2\cos\theta$



$\int \sqrt{4-x^2}\,dx$

$= \int \sqrt{4 - 4\sin^2\theta}\; 2\cos\theta\,d\theta$

$= \int (2\cos\theta)(2\cos\theta)\,d\theta$

$= 4\int \cos^2\theta\,d\theta = \frac{4}{2}\int 1 + \cos 2\theta\,d\theta$

$= 2\left(\theta + \frac{\sin 2\theta}{2}\right) + c$

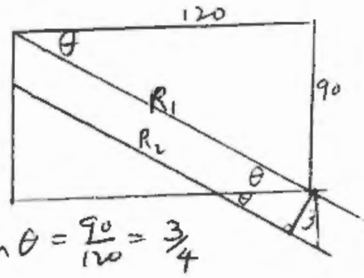
$= 2\sin^{-1}\left(\frac{x}{2}\right) + 2\sin\theta\cos\theta + c$

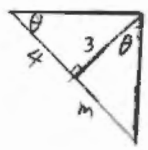
$= 2\sin^{-1}\left(\frac{x}{2}\right) + \cancel{2}\left(\frac{x}{\cancel{2}}\right)\frac{\sqrt{4-x^2}}{2} + c$

$= \frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\left(\frac{x}{2}\right) + c$ #

b) $R_1 = \sqrt{120^2 + 90^2} = 150$ m #

ii)





$\tan\theta = \frac{90}{120} = \frac{3}{4}$

$\tan\theta = \frac{3}{4} = \frac{m}{3}$

$4m = 9$

$m = 2.25$

$\therefore R_2 = 150 - 2.25 - 4$
$= 143.75$ m #

iii) Cut the rectangle into 2 halves (triangles). Let n be the last row in the triangle

$R_n = a + (n-1)d = 150 - 6.25(n-1) \geq 0$

$150 \geq 6.25(n-1)$

$24 \geq n-1$

$n < 25 \quad \therefore n = 24$

Total number of rows of vines

$= 23 + 24 = 47$ #